

DISCRETENESS OF LOG DISCREPANCIES OVER LOG CANONICAL TRIPLES ON A FIXED PAIR

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ABSTRACT. For a fixed pair and fixed exponents, we prove the discreteness of log discrepancies over all log canonical triples formed by attaching a product of ideals with given exponents.

1. INTRODUCTION

The log minimal model program (LMMP) is a program to find a good representative in each birational equivalence class of varieties by comparing the log canonical divisors. The log discrepancy, appearing in the relative log canonical divisor, is hence a fundamental invariant in the LMMP. For a triple $(X, \Delta, \mathfrak{a})$, the *log discrepancy* $a_E(X, \Delta, \mathfrak{a})$ is attached to each divisor E over X . The minimum of those $a_E(X, \Delta, \mathfrak{a})$ with E mapped onto a subset Z of X is called the *minimal log discrepancy* at η_Z and denoted by $\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$. Refer to Section 2 for the precise definitions.

Shokurov conjectures the ascending chain condition (ACC) [10], [11, Conjecture 4.2] of the set of the minimal log discrepancies of all pairs with given coefficients in fixed dimension. Its importance is recognised in his reduction [12] of the termination of flips (in the relatively projective case) to this ACC and the lower semi-continuity of minimal log discrepancies. The main theorem of this paper is the *discreteness* of log discrepancies over log canonical triples on a fixed pair. Let \mathcal{D}_X denote the set of divisors over X .

Theorem 1.1. *Let (X, Δ) be a pair and $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{a_E(X, \Delta, \prod_{j=1}^k \mathfrak{a}_j^{r_j}) \mid \mathfrak{a}_j \subset \mathcal{O}_X, E \in \mathcal{D}_X, (X, \Delta, \prod_{j=1}^k \mathfrak{a}_j^{r_j}) \text{ lc at } \eta_{c_X(E)}\}$$

is discrete in \mathbb{R} , where $c_X(E)$ is the centre of E on X .

Note that it is trivial in the case of rational boundary and exponents. The condition of log canonicity is necessary, see Remark 5.1.

Theorem 1.1 asserts a special case of Shokurov's ACC conjecture.

Theorem 1.2. *Let (X, Δ) be a pair and $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{\text{mld}_{\eta_Z}(X, \Delta, \prod_{j=1}^k \mathfrak{a}_j^{r_j}) \mid \mathfrak{a}_j \subset \mathcal{O}_X, Z \subset X\}$$

is finite.

Theorem 1.2 follows from Theorem 1.1 immediately since the minimal log discrepancies in Theorem 1.2 are bounded from above by the maximum of $\text{mld}_x(X, \Delta)$ for all $x \in X$.

We prove Theorem 1.1 in Section 4 by developing the study [3], [4], [5], [9] of the ACC for *log canonical thresholds* due to de Fernex, Ein, Mustařă and Kollár. We use their construction of a *generic limit* (W, \mathfrak{a}) , reviewed in Section 3, from a collection of bounded singularities W_i and ideals \mathfrak{a}_i . W is the spectrum of a complete local ring over an extension of the ground field. They showed the equivalence of the log canonicity of (W, \mathfrak{a}) and general (W_i, \mathfrak{a}_i) in order to obtain the ACC for log canonical thresholds on bounded singularities. We apply this equivalence to small perturbations of the exponents in \mathfrak{a} . It brings the boundedness of the orders appearing in the expression of $a_{F_i}(W_i, \mathfrak{a}_i)$, leading Theorem 1.1.

Several extensions of Theorems 1.1, 1.2 and relevant remarks are given in Section 5. For example, applying to locally complete intersection (lci) singularities, we obtain the following in Corollary 5.4.

Theorem 1.3. *Fix an integer d and $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{\text{mld}_{\eta_Z}(X, \prod_{j=1}^k \mathfrak{a}_j^{r_j}) \mid X \text{ lci, } \dim X \leq d, \mathfrak{a}_j \subset \mathcal{O}_X, Z \subset X\}$$

is finite.

We work over an algebraically closed field k of characteristic zero.

2. LOG DISCREPANCIES

A *pair* (X, Δ) consists of a normal variety X and a *boundary* Δ , that is, an effective \mathbb{R} -divisor such that $K_X + \Delta$ is an \mathbb{R} -Cartier \mathbb{R} -divisor. We treat a *triple* $(X, \Delta, \mathfrak{a})$ by attaching a formal product $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ of finitely many coherent ideal sheaves \mathfrak{a}_j with real exponents $r_j \in \mathbb{R}_{\geq 0}$. An *extraction* of X is a normal variety X' with a proper birational morphism $\varphi: X' \rightarrow X$. A prime divisor E on such an extraction X' is called a divisor *over* X , and the image $\varphi(E)$ on X is called the *centre* of E on X and denoted by $c_X(E)$. We denote by \mathcal{D}_X the set of divisors over X . We define the *log discrepancy* of E with respect to the triple $(X, \Delta, \mathfrak{a})$ as

$$a_E(X, \Delta, \mathfrak{a}) := 1 + \text{ord}_E(K_{X'} - \varphi^*(K_X + \Delta)) - \text{ord}_E \mathfrak{a},$$

where $\text{ord}_E \mathfrak{a} := \sum_j r_j \text{ord}_E \mathfrak{a}_j$ for $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$. The triple $(X, \Delta, \mathfrak{a})$ is said to be *log canonical* (lc), *Kawamata log terminal* (klt) if $a_E(X, \Delta, \mathfrak{a}) \geq 0, > 0$ respectively for all $E \in \mathcal{D}_X$, and said to be *canonical*, *terminal* if $a_E(X, \Delta, \mathfrak{a}) \geq 1, > 1$ respectively for all exceptional $E \in \mathcal{D}_X$. Let Z be an irreducible closed subset of X and η_Z its generic point. The *minimal log discrepancy* $\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$ at η_Z is the infimum of $a_E(X, \Delta, \mathfrak{a})$ for all $E \in \mathcal{D}_X$ with centre Z . It is either a non-negative real number or $-\infty$. The log canonicity of $(X, \Delta, \mathfrak{a})$ about η_Z is equivalent to $\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) \geq 0$. We say that $E \in \mathcal{D}_X$ *computes* $\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$ if $c_X(E) = Z$ and $a_E(X, \Delta, \mathfrak{a}) = \text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$ (or negative when $\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) = -\infty$). We are often reduced to the case when Z is a closed point since $\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) = \text{mld}_z(X, \Delta, \mathfrak{a}) - \dim Z$ for general $z \in Z$, see [1, Proposition 2.1].

3. GENERIC LIMITS

The generic limit is a limit of ideals in a fixed local ring. It was constructed first by de Fernex and Mustařă in [5] using ultraproducts, and the construction was then simplified by Kollár in [9]. It is clearly exposed in [3, Section 4], [4, Section 3].

Set $R = k[[x_1, \dots, x_N]]$ with maximal ideal $\tilde{\mathfrak{m}}$. We fix integers m and k . For every l , let \mathcal{H}_l be the Hilbert scheme parametrising ideals in R containing $\tilde{\mathfrak{m}}^l$. Let \mathcal{G} be the parameter space for ideals in R generated by polynomials in $\tilde{\mathfrak{m}}$ of degree $\leq m$. Set $\mathcal{Z}_l = \mathcal{G} \times (\mathcal{H}_l)^k$. We have a natural surjective map $t_l: \mathcal{Z}_l \rightarrow \mathcal{Z}_{l-1}$, and by generic flatness, there exists a stratification of \mathcal{Z}_l such that the restriction of t_l on each stratum is a morphism.

We define a generic limit of the collection $\{(\tilde{\mathfrak{p}}_i; \tilde{\mathfrak{a}}_{i1}, \dots, \tilde{\mathfrak{a}}_{ik})\}_{i \in I}$ of $(k+1)$ -tuples of ideals in R indexed by an infinite set I , where $\tilde{\mathfrak{p}}_i$ are generated by polynomials in $\tilde{\mathfrak{m}}$ of degree $\leq m$. One can construct locally closed irreducible subsets $Z_l^\circ \subset \mathcal{Z}_l$ such that

- (i) t_l induces a dominant morphism $Z_l^\circ \rightarrow Z_{l-1}^\circ$,
- (ii) $I_l := \{i \in I \mid (\tilde{\mathfrak{p}}_i; \tilde{\mathfrak{a}}_{i1} + \tilde{\mathfrak{m}}^l, \dots, \tilde{\mathfrak{a}}_{ik} + \tilde{\mathfrak{m}}^l) \in Z_l^\circ\}$ is infinite,
- (iii) the set of points in \mathcal{Z}_l indexed by I_l is dense in Z_l° .

We take the union $K = \bigcup_l k(Z_l^\circ)$ of the function fields by the inclusions $k(Z_{l-1}^\circ) \subset k(Z_l^\circ)$. For each l , the morphism $\text{Spec } K \rightarrow Z_l^\circ \subset \mathcal{Z}_l$ corresponds to a $(k+1)$ -tuple $(\tilde{\mathfrak{p}}(l); \tilde{\mathfrak{a}}_1(l), \dots, \tilde{\mathfrak{a}}_k(l))$ of ideals in $R_K = R \otimes_k K$. Then there exists a $(k+1)$ -tuple $(\tilde{\mathfrak{p}}; \tilde{\mathfrak{a}}_1, \dots, \tilde{\mathfrak{a}}_k)$ of ideals in R_K such that $\tilde{\mathfrak{p}}(l) = \tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{a}}_j(l) = \tilde{\mathfrak{a}}_j + \tilde{\mathfrak{m}}_K^l$, where $\tilde{\mathfrak{m}}_K = \tilde{\mathfrak{m}}R_K$. This $(\tilde{\mathfrak{p}}; \tilde{\mathfrak{a}}_1, \dots, \tilde{\mathfrak{a}}_k)$ is a *generic limit* of our collection $\{(\tilde{\mathfrak{p}}_i; \tilde{\mathfrak{a}}_{i1}, \dots, \tilde{\mathfrak{a}}_{ik})\}_{i \in I}$.

We set $W_i = \text{Spec } R/\tilde{\mathfrak{p}}_i$, $W = \text{Spec } R_K/\tilde{\mathfrak{p}}$ with closed points $o_i \in W_i$, $o \in W$, and $\mathfrak{m}_i = \tilde{\mathfrak{m}}/\tilde{\mathfrak{p}}_i$, $\mathfrak{m} = \tilde{\mathfrak{m}}_K/\tilde{\mathfrak{p}}$. We suppose $\tilde{\mathfrak{p}}_i \subset \tilde{\mathfrak{a}}_{ij}$, then $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{a}}_j$ and write $\mathfrak{a}_{ij} = \tilde{\mathfrak{a}}_{ij}/\tilde{\mathfrak{p}}_i$, $\mathfrak{a}_j = \tilde{\mathfrak{a}}_j/\tilde{\mathfrak{p}}$.

We compare log discrepancies over W and W_i . The notions in Section 2 are extended to the spectra of our complete local rings by the existence of their log resolutions due to Temkin in [13] after Hironaka. This extension is discussed in [4], [5] by de Fernex, Ein and Mustařă. The following proposition associates the minimal log discrepancy of the generic limit to those of the \mathfrak{m}_i -adic approximations of the original pairs. Proposition 3.2 is a consequence of the basic fact that for a family of pairs, the minimal log discrepancy is constant on an open subfamily. The corresponding statement for log canonical thresholds is [3, Proposition 4.4] or [4, Proposition 3.3]. Our (iii) is stronger, but it just needs the extra condition $l_E > \text{ord}_E \mathfrak{a}_j$ for any j .

Definition 3.1. A subset J_l of I_l is said to be *dense* if J_l is infinite and the set of points in \mathcal{Z}_l indexed by J_l is dense in Z_l° .

Proposition 3.2. Suppose that W_i has log terminal singularities. Then

- (i) W has log terminal singularities.
- (ii) For each l , there exists a dense subset I_l° of I_l such that

$$\text{mld}_o(W, \prod_j (\mathfrak{a}_j + \mathfrak{m}^l)^{r_j}) = \text{mld}_{o_i}(W_i, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}_i^l)^{r_j})$$

for all $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$ and all $i \in I_l^\circ$.

- (iii) Fix $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$ and $E \in \mathcal{D}_W$ computing $\text{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j})$. Then there exist an integer l_E , a dense subset I_l^E of I_l° for each $l \geq l_E$, and $E_i \in \mathcal{D}_{W_i}$ computing $\text{mld}_{o_i}(W_i, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}_i^l)^{r_j})$ for each $i \in I_l^E$, such that

$$\text{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j}) = \text{mld}_{o_i}(W_i, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}_i^l)^{r_j}),$$

$$\text{ord}_E \mathfrak{a}_j = \text{ord}_E(\mathfrak{a}_j + \mathfrak{m}^l) = \text{ord}_{E_i}(\mathfrak{a}_{ij} + \mathfrak{m}_i^l) = \text{ord}_{E_i} \mathfrak{a}_{ij} < l,$$

for all $i \in I_l^E$ with $l \geq l_E$.

Remark 3.3. One can choose $l_{E(s)}$ and $I_l^{E(s)}$ in Proposition 3.2(iii) commonly for a finite collection $\{(r_1(s), \dots, r_k(s); E(s))\}_s$.

We shall use the effective ideal-adic semi-continuity of log canonicity due to Kollár, and de Fernex, Ein and Mustařă. They applied it to the ACC for log canonical thresholds on bounded singularities.

Theorem 3.4 ([9, Theorem 32], [3, Theorem 1.4]). *Let $W = \widehat{\text{Spec } \mathcal{O}_{X,x}}$ with closed point o for some log canonical singularity $x \in X$, and $\mathfrak{a}_1, \dots, \mathfrak{a}_k, \mathfrak{b}_1, \dots, \mathfrak{b}_k \subset \mathcal{O}_W$, $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$. Suppose $\text{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j}) = 0$ and it is computed by $E \in \mathcal{D}_W$. If $\mathfrak{a}_j + \mathfrak{p}_j = \mathfrak{b}_j + \mathfrak{p}_j$ for every j , where $\mathfrak{p}_j = \{f \in \mathcal{O}_W \mid \text{ord}_E f > \text{ord}_E \mathfrak{a}_j\}$, then $\text{mld}_o(W, \prod_j \mathfrak{b}_j^{r_j}) = 0$.*

Corollary 3.5. *If $\text{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j}) = 0$ in Proposition 3.2(iii), then $\text{mld}_{o_i}(W_i, \prod_j \mathfrak{a}_{ij}^{r_j}) = 0$ for $i \in I_l^E$.*

4. DISCRETENESS

The purpose of this section is to prove Theorem 1.1. The theorem is reduced to the case when X has \mathbb{Q} -factorial terminal singularities by the existence of a \mathbb{Q} -factorial terminal extraction $\varphi: X' \rightarrow X$ with $\Delta' \geq 0$ for $K_{X'} + \Delta' = \varphi^*(K_X + \Delta)$, thanks to [2]. Then we may assume $\Delta = 0$ by forcing $\prod_j \mathfrak{a}_{ij}^{r_j}$ to absorb Δ . Moreover, we may consider only the log discrepancies of divisors whose centres are closed points. Hence it suffices to prove the following theorem.

Theorem 4.1. *Let X be a variety with log terminal singularities and $a, r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$. Let I be an infinite set indexing $a_i = a_{E_i}(X, \prod_j \mathfrak{a}_{ij}^{r_j}) \leq a$ with $\mathfrak{a}_{i1}, \dots, \mathfrak{a}_{ik} \subset \mathcal{O}_X$, $E_i \in \mathcal{D}_X$ such that $x_i = c_X(E_i)$ is a closed point at which $(X, \prod_j \mathfrak{a}_{ij}^{r_j})$ is log canonical. Then there exists an infinite subset $I^\circ \subset I$ such that a_i is constant for $i \in I^\circ$.*

Since X is covered by finitely many affine open subvarieties, we can fix integers N and m so that for each $i \in I$ there exists an ideal $\tilde{\mathfrak{p}}_i$ in $R = k[[x_1, \dots, x_N]]$ generated by polynomials of degree $\leq m$ which satisfies $\widehat{\mathcal{O}_{X,x_i}} \simeq R/\tilde{\mathfrak{p}}_i$. We apply the generic limit construction in Section 3 to the collection $\{(\tilde{\mathfrak{p}}_i; \tilde{\mathfrak{a}}_{i1}, \dots, \tilde{\mathfrak{a}}_{ik})\}_{i \in I}$; here we let \mathfrak{a}_{ij} denote also the image in $R/\tilde{\mathfrak{p}}_i$ of $\mathfrak{a}_{ij} \widehat{\mathcal{O}_{X,x_i}}$ by abuse of notation, and define $\tilde{\mathfrak{a}}_{ij}$ as the inverse image in R of \mathfrak{a}_{ij} . The generic limit $(\tilde{\mathfrak{p}}; \tilde{\mathfrak{a}}_1, \dots, \tilde{\mathfrak{a}}_k)$ is defined in some $R_K = R \otimes_k K$. We follow the notation in Section 3. We have $a_i = a_{F_i}(W_i, \prod_j \mathfrak{a}_{ij}^{r_j})$ for $F_i = E_i \times_X W_i$, and $(W_i, \prod_j \mathfrak{a}_{ij}^{r_j})$ is log canonical. By Proposition 3.2(ii), $(W, \prod_j \mathfrak{a}_j^{r_j})$ is also log canonical.

We shall find perturbations of the exponents r_j preserving the log canonicity. Set $r_0 = 1$. By permutation, we may assume that $r_0, \dots, r_{k'}$ for some $0 \leq k' \leq k$ form a basis of the \mathbb{Q} -vector space spanned by r_0, \dots, r_k . We write $r_j = \sum_{j'=0}^{k'} q_{jj'} r_{j'}$ with $q_{jj'} \in \mathbb{Q}$, then $\prod_j \mathfrak{a}_j^{r_j} = \prod_{j'=0}^{k'} (\prod_j \mathfrak{a}_j^{q_{jj'}})^{r_{j'}}$ formally. We put $\mathfrak{b}_{j'} := \prod_j \mathfrak{a}_j^{q_{jj'}}$ and $\mathfrak{b}_{ij'} := \prod_j \mathfrak{a}_{ij}^{q_{jj'}}$. Setting $s_0 = 1$, for $\varepsilon > 0$ we define the finite set

$$S_\varepsilon := \{(s_1, \dots, s_k) \mid s_j = \sum_{j'=0}^{k'} q_{jj'} s_{j'} \ \forall j, |s_{j'} - r_{j'}| = \varepsilon \ 1 \leq \forall j' \leq k'\}.$$

Lemma 4.2. *There exist ε , l and an dense subset $J_l \subset I_l$ such that all $s_j \geq 0$ and $(W_i, \prod_j \mathfrak{a}_{ij}^{s_j})$ is log canonical for any $(s_j) \in S_\varepsilon$ and $i \in J_l$.*

Proof. First we find ε such that all $s_j \geq 0$ and $(W, \prod_j \mathfrak{a}_j^{s_j})$ is log canonical for any $(s_j) \in S_\varepsilon$. Indeed, if $E \in \mathcal{D}_W$ has $a_E(W, \prod_j \mathfrak{a}_j^{r_j}) = 0$, then $a_E(W) = \sum_{j'=0}^{k'} r_{j'} \text{ord}_E \mathfrak{b}_{j'}$, and thus $\text{ord}_E \mathfrak{b}_{j'} = 0$ for $1 \leq j' \leq k'$ by the \mathbb{Q} -linear independence of $r_{j'}$. This means that the log discrepancy of E remains zero when we perturb the exponents $r_1, \dots, r_{k'}$ in $\prod_{j'} \mathfrak{b}_{j'}^{r_{j'}} (= \prod_j \mathfrak{a}_j^{r_j})$. Hence on a fixed log resolution of $(W, \prod_j \mathfrak{a}_j^{r_j})$, every divisor remains to have non-negative log discrepancy by sufficiently small such perturbation, which guarantees the existence of the required ε .

For each $s = (s_j) \in S_\varepsilon$, we fix $t_s \geq 0$ such that $\text{mld}_o(W, \prod_j \mathfrak{a}_j^{s_j} \mathfrak{m}_i^{t_s}) = 0$. By Corollary 3.5 and Remark 3.3, we obtain l and $J_l \subset I_l$ such that $\text{mld}_{o_i}(W_i, \prod_j \mathfrak{a}_{ij}^{s_j} \mathfrak{m}_i^{t_s}) = 0$ for any $s \in S_\varepsilon$ and $i \in J_l$, meaning the log canonicity of $(W_i, \prod_j \mathfrak{a}_{ij}^{s_j})$. q.e.d.

Lemma 4.3. $\sum_{j'=1}^{k'} |\text{ord}_{F_i} \mathfrak{b}_{ij'}| \leq \varepsilon^{-1} a_i$ for $i \in J_l$.

Proof. We choose $s_{ij'} = r_{j'} \pm \varepsilon$ for $1 \leq j' \leq k'$ so that $\text{ord}_{F_i} \mathfrak{b}_{ij'} / (s_{ij'} - r_{j'}) \geq 0$, and extend $(s_{ij'})$ to the k -tuple $(s_{ij}) \in S_\varepsilon$. Then by Lemma 4.2, $0 \leq a_{F_i}(W_i, \prod_j \mathfrak{a}_{ij}^{s_{ij}}) = a_i - \varepsilon \sum_{j'=1}^{k'} |\text{ord}_{F_i} \mathfrak{b}_{ij'}|$. q.e.d.

Fix a positive integer n such that nK_X is a Cartier divisor and $nq_{jj'} \in \mathbb{Z}$ for all j, j' . We define the finite set

$$A := [0, a] \cap \left(\frac{1}{n} \mathbb{Z} + \left\{ \sum_{j'=1}^{k'} r_{j'} m_{j'} \mid \sum_{j'=1}^{k'} |m_{j'}| \leq \varepsilon^{-1} a, m_{j'} \in \frac{1}{n} \mathbb{Z} \forall j' \right\} \right).$$

Then by Lemma 4.3, $a_i = a_{F_i}(W_i, \prod_{j'=0}^{k'} \mathfrak{b}_{ij'}^{r_{j'}}) = a_{F_i}(W_i, \mathfrak{b}_{i0}) - \sum_{j'=1}^{k'} r_{j'} \text{ord}_{F_i} \mathfrak{b}_{ij'} \in A$ for $i \in J_l$. Theorem 4.1, and Theorem 1.1, are therefore completed.

5. EXTENSIONS

First we remark the need of log canonicity in Theorem 1.1.

Remark 5.1. Consider a non-lc pair $(\mathbb{A}^2, (1+r)l)$ where $r > 0$ is irrational and l is a line. Let E_1 be the exceptional divisor of the blow-up of \mathbb{A}^2 at a point on l , and define E_p inductively as the exceptional divisor of the blow-up at the intersection of E_{p-1} and the strict transform of l . Let $E_{p,0} = E_p$ and define $E_{p,q}$ inductively as the exceptional divisor of the blow-up at a general point on $E_{p,q-1}$. Then $a_{E_{p,q}}(\mathbb{A}^2, (1+r)l) = q - pr$. The set of these log discrepancies is dense in \mathbb{R} .

The generic limit construction is applicable to bounded singularities in the sense [4] of de Fernex, Ein and Mustařă. We say that a collection $\{x_i \in X_i\}_i$ of singularities is *bounded* if there exist m and N such that for each i there exists an ideal $\tilde{\mathfrak{p}}_i$ in $R = k[[x_1, \dots, x_N]]$ generated by polynomials of degree $\leq m$ which satisfies $\widehat{\mathcal{O}}_{X_i, x_i} \simeq R/\tilde{\mathfrak{p}}_i$. Theorem 1.1 is formulated for such a collection.

Theorem 5.2. *Let \mathcal{X} be a collection of varieties with bounded log terminal singularities, and $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{a_E(X, \prod_{j=1}^k \mathfrak{a}_j^{r_j}) \mid X \in \mathcal{X}, \mathfrak{a}_j \subset \mathcal{O}_X, E \in \mathcal{D}_X, (X, \prod_{j=1}^k \mathfrak{a}_j^{r_j}) \text{ lc at } \eta_{c_X(E)}\}$$

is discrete.

We apply it to the minimal log discrepancies of Gorenstein singularities.

Corollary 5.3. *Let \mathcal{X} be a collection of varieties with bounded normal Gorenstein singularities, and $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{\text{mld}_{\eta_Z}(X, \prod_{j=1}^k \mathfrak{a}_j^{r_j}) \mid X \in \mathcal{X}, \mathfrak{a}_j \subset \mathcal{O}_X, Z \subset X\}$$

is finite.

This follows from the boundedness [7, Theorem 2.2] of the minimal log discrepancies of Gorenstein singularities with bounded embedding dimensions. Note that even for Gorenstein log canonical singularities, [7, Theorem 2.2] holds by its proof, and Proposition 3.2(ii), (iii) hold since [4, Appendix B] is unnecessary.

We have a further application to lci singularities.

Corollary 5.4. *Fix an integer d and $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$. Then the set*

$$\{a_E(X, \prod_{j=1}^k \mathfrak{a}_j^{r_j}) \mid X \text{ lci}, \dim X \leq d, \mathfrak{a}_j \subset \mathcal{O}_X, E \in \mathcal{D}_X, (X, \prod_{j=1}^k \mathfrak{a}_j^{r_j}) \text{ lc at } \eta_{c_X(E)}\}$$

is discrete.

Corollary 5.4, and Theorem 1.3, follow from inversion of adjunction [6] on lci varieties. We also use its consequence that an lci log canonical singularity of dimension d has embedding dimension $\leq 2d$, see [3, Proposition 6.3].

On the other hand, Shokurov's ACC conjecture is generalised to the case when the exponents vary in a fixed set satisfying the descending chain condition (DCC). Mustařa observed that an effective ideal-adic semi-continuity for minimal log discrepancies implies the generalised ACC on a fixed pair (see [8, Remark 1.5.1]). This semi-continuity is known in the klt case [8, Theorem 1.6], and for example, we can prove the following.

Proposition 5.5. *Let (X, Δ) be a pair with rational Δ , and R a subset of $\mathbb{R}_{\geq 0}$ satisfying the DCC. Suppose that any accumulation point of R is irrational. Then the set*

$$\{\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}^r) \mid \mathfrak{a} \subset \mathcal{O}_X, r \in R, Z \subset X\}$$

satisfies the ACC.

Proof. As in the beginning of Section 4, we are reduced to the case of terminal X , and we want the stability of any non-decreasing sequence of $a_i = \text{mld}_{x_i}(X, \Delta, \mathfrak{a}_i^{r_i}) \geq 0$ with $r_i \in R$, x_i closed point, where $i \in \mathbb{N}$. By passing to a subsequence, we may assume that \mathfrak{a}_i is non-trivial at x_i . Then r_i are bounded by the maximum b of $\text{mld}_x(X, \Delta)$ for all $x \in X$, hence we may further assume that $\{r_i\}_i$ is a non-decreasing sequence which has a limit r . If $r \in \mathbb{Q}$, then $r_i = r$ for large i by the assumption on R , and the stability is trivial. Henceforth we assume $r \notin \mathbb{Q}$. As in Section 4, we construct a generic limit $o \in W$, $\mathfrak{d}, \mathfrak{a}$ of $o_i \in W_i$, $\mathfrak{d}_i, \mathfrak{a}_i$ with $a_i = \text{mld}_{o_i}(W_i, \mathfrak{d}_i \mathfrak{a}_i^{r_i})$, where \mathfrak{d}_i is an ideal with fixed rational exponent, corresponding to Δ . We fix $F_i \in \mathcal{D}_{W_i}$ computing $\text{mld}_{o_i}(W_i, \mathfrak{d}_i \mathfrak{a}_i^{r_i})$, that is,

$$(1) \quad a_i = a_{F_i}(W_i, \mathfrak{d}_i \mathfrak{a}_i^{r_i}) + (r - r_i) \text{ord}_{F_i} \mathfrak{a}_i.$$

By Proposition 3.2 or [4, Corollary 3.4], $(W, \mathfrak{d} \mathfrak{a}^r)$ is log canonical. If $E \in \mathcal{D}_W$ has $a_E(W, \mathfrak{d} \mathfrak{a}^r) = 0$, then $\text{ord}_E \mathfrak{a} = 0$ by $r \notin \mathbb{Q}$. Thus we can find $t > 0$ such that

$(W, \mathfrak{d}\mathfrak{a}^{r+t})$ is log canonical as in the proof of Lemma 4.2. We take $t' \geq 0$ such that $\text{mld}_o(W, \mathfrak{d}\mathfrak{a}^{r+t'}\mathfrak{m}_i^{t'}) = 0$. Then Corollary 3.5 shows $\text{mld}_{o_i}(W_i, \mathfrak{d}_i\mathfrak{a}_i^{r+t'}\mathfrak{m}_i^{t'}) = 0$ for infinitely many i . In particular,

$$(2) \quad t \text{ord}_{F_i} \mathfrak{a}_i \leq a_{F_i}(W_i, \mathfrak{d}_i\mathfrak{a}_i^r) \leq a_i \leq b.$$

Theorem 1.1 and (2) imply the finiteness of possible choices for $a_{F_i}(W_i, \mathfrak{d}_i\mathfrak{a}_i^r)$ and $\text{ord}_{F_i} \mathfrak{a}_i$ for such i . Hence they are constant for infinitely many i . Now (1) provides the constancy of a_i and r_i for large i . q.e.d.

Remark 5.6. When $\mathfrak{d}_i = \mathcal{O}_{W_i}$ and $r \notin \mathbb{Q}$ in the proof, one can further prove $a_i = a := \text{mld}_o(W, \mathfrak{a}^r)$ for large i . Indeed, we may assume $r_i = r$, and $a = \text{mld}_{o_i}(W_i, (\mathfrak{a}_i + \mathfrak{m}_i^l)^r) \geq a_i$ for some $l > t^{-1}b$ by Proposition 3.2(iii). Hence with (2), we have $a \geq a_i = a_{F_i}(W_i, \mathfrak{a}_i^r) = a_{F_i}(W_i, (\mathfrak{a}_i + \mathfrak{m}_i^l)^r) \geq a$, meaning $a_i = a$.

Remark 5.6 is a special case of the following conjecture. The corresponding statement for log canonical thresholds is [4, Corollary 3.4].

Conjecture 5.7. *With the notation in Section 3, we suppose that W_i has log terminal singularities and fix $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$. Let*

$$J := \{i \in I \mid \text{mld}_{o_i}(W_i, \prod_j \mathfrak{a}_{ij}^{r_j}) = \text{mld}_o(W, \prod_j \mathfrak{a}_j^{r_j})\}.$$

Then $I_l \cap J$ is dense for each l .

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